

Entropy of a Differentiable Map

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1. INTRODUCTION

The "entropy conjecture" asserts that if T is a differentiable map of a compact manifold X , the topological entropy (cf. [1]) h of T is at least as great as $\log \lambda$, where λ denotes the spectral radius of T^* , the map on de Rham cohomology induced by T . Various results related to the entropy conjecture have been obtained [1, 3, 4-6]. On the other hand, it is known that no such conjecture is true for maps that are Lipschitz rather than differentiable [4, 5].

We have not been able to prove the entropy conjecture in full generality; however, we establish several closely related inequalities. In Section 2, numbers h_2 and h_3 that are related to Hausdorff measure and have properties similar to those of h are defined. It is shown that $h_3 \leq h$, and under certain conditions $h_2 = h_3$. In Section 3, another number h_1 is defined in terms of the derivative DT^n and $\log \lambda \leq h_1$ is demonstrated. In Section 4 it is proved that $h_1 \leq h_2$ and thus we always have $\log \lambda \leq h_1 \leq h_2$ and $h_3 \leq h$. Thus what is lacking to prove the entropy conjecture in full generality is the inequality $h_2 \leq h_3$, which we have only been able to prove under special conditions.*

2. h_2 AND h_3

It is assumed that X is a compact manifold of dimension m and that X has a Riemannian metric; d denotes the distance function that defines $d(x, y)$ to be the greatest lower bound of the lengths of the arcs connecting points x and y of X . Define a sequence of distance functions d_1, d_2, \dots by $d_n(x, y) = \max\{d(T^k(x), T^k(y)) : 0 \leq k \leq n\}$ and let $D_n(x, \epsilon) = \{y \in X : d_n(x, y) \leq \epsilon\}$. If $r(n, \epsilon)$ denotes the minimum cardinality of a set $\{x_1, \dots, x_r\}$ such that $X = \bigcup D_n(x_i, \epsilon)$, the results of Bowen [1] show that

$$h = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (1/n) \log r(n, \epsilon). \quad (2.1)$$

* M. Misiurewicz and W. Szlenk have constructed a smooth map for which $h_1 > h$, hence $h_2 > h_3$. ("Entropy of Piecewise Monotone Mappings," University of Warsaw Preprint.)

Let Ω_m be the volume of a Euclidean sphere of radius 1. For any $\epsilon > 0$ and $n = 1, 2, \dots$ let $M(n, \epsilon)$ be the greatest lower bound of $\Omega_m \sum_{i=1}^p \epsilon_i^m$, where $X = \bigcup_{i=1}^p D_n(x_i, \epsilon_i)$ and $\epsilon_i \leq \epsilon$ for $i = 1, \dots, p$. We define

$$h_3 = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (1/n) \log M(n, \epsilon).$$

It is obvious that

$$M(n, \epsilon) \leq r(n, \epsilon) \Omega_m \epsilon^m,$$

hence (2.1) and the definition of h_3 give our first inequality,

$$h_3 \leq h. \quad (2.2)$$

The quantity h_2 is defined by taking limits in the opposite order, that is, $h_2 = \lim_{n \rightarrow \infty} (1/n) \lim_{\epsilon \rightarrow 0} \log M(n, \epsilon)$. There is no difficulty about the existence of the first limit. In fact, if we denote the m -dimensional Hausdorff measure of a subset A of X , with respect to the metric d_n by $M_n(A)$, then the definition of Hausdorff measure shows that $M_n(X) = \lim_{\epsilon \rightarrow 0} M(n, \epsilon)$.

Since $M_n(X) \geq M(n, \epsilon) \geq M(n, \delta)$ if $\epsilon \leq \delta$, one always has $h_2 \geq h_3$. One would like to be able to show that $h_2 = h_3$; however, we have only been able to show this under the conditions of the proposition below.

PROPOSITION 2.1. *Suppose that there is a sequence a_1, a_2, \dots of positive constants such that $\lim(a_n)^{1/n} = 1$ as $n \rightarrow \infty$ and for all x in X ,*

$$M_n(D_n(x, \epsilon)) \leq a_n \epsilon^m.$$

Then

$$h_2 = h_3.$$

The proof just depends on observing that if $X = \bigcup_{i=1}^p D_n(x_i, \epsilon_i)$, then

$$M_n(X) \leq \sum_{i=1}^p M_n(D_n(x_i, \epsilon_i)) \leq a_n \Omega_m \sum_{i=1}^p \epsilon_i^m;$$

hence

$$M_n(X) \leq a_n M(n, \epsilon) \quad \text{for every } n \text{ and } \epsilon.$$

The conclusion follows easily.

3. THE MAPS T^* AND E_T

Let ω_1 and ω_2 be complex-valued p -forms defined on X . We define for x in X , $(\omega_1, \omega_2)(x) = *(\omega_1 \wedge *\bar{\omega}_2)(x)$, where $*$ is the Hodge operator and $-$ denotes complex conjugation. Also,

$$|\omega|(x) = (\omega, \omega)(x)^{1/2}.$$

Then (cf. [7, Chap. 6])

$$\langle \omega_1, \omega_2 \rangle = \int_X (\omega_1, \omega_2)(x) dV$$

defines an inner product on the space of p -forms and the norm

$$\|\omega\|_q = \left(\int_X |\omega(x)|^q dV \right)^{1/q}$$

is defined for $q \geq 1$. Also define $\|\omega\|_\infty = \sup\{|\omega(x)| : x \in X\}$. Here dV is just the volume element associated to the metric, that is, $dV = *1$. The completion of the space of forms (of dimension $p = 0, \dots, m$) with respect to $\|\cdot\|_q$ is denoted by L^q , where it is understood that the inner product in L^2 is defined to make homogeneous forms of different dimensions orthogonal.

Let H denote the orthogonal projection from L^2 to the harmonic forms (cf. [7, Chap. 6]). Since $L^2 \subset L^1$ it makes sense to ask if H is bounded with respect to $\|\cdot\|_1$ and as is well known, the answer is yes.

PROPOSITION 3.1. *H is bounded with respect to $\|\cdot\|_1$.*

Proof. Let $\alpha_1, \dots, \alpha_r$ be an orthonormal basis for the harmonic forms. Then $H\omega = \sum_{j=1}^r \langle \omega, \alpha_j \rangle \alpha_j$, so

$$|H\omega|(x) \leq \sum_{j=1}^r |\langle \omega, \alpha_j \rangle| |\alpha_j|(x) \quad \text{and} \quad \|H\omega(x)\|_1 \leq \sum_{j=1}^r |\langle \omega, \alpha_j \rangle| \|\alpha_j\|_1.$$

But,

$$|\langle \omega, \alpha_j \rangle| \leq \int_X |\omega|(x) |\alpha_j|(x) dV \leq \|\alpha_j\|_\infty \|\omega\|_1,$$

so

$$\|H\|_1 \leq \sum_{j=1}^r \|\alpha_j\|_\infty \|\alpha_j\|_1.$$

Now let E_T denote the map on forms and DT the tangent map induced by T . Then if the map on antisymmetric p -tuples of tangent vectors corresponding to DT is denoted by $D_p T$, one has

$$(E_T \omega)(X_1, \dots, X_p) = \omega(DT(X_1), \dots, DT(X_p)) = \omega(D_p T(X_1, \dots, X_p)).$$

Define $\theta_n^p(x) = \|D_p T^n(x)\|$, where the norm of $D_p T$ on the antisymmetric p -tuples of tangent vectors is determined by the Riemannian metric on X . Then $|(E_T^n \alpha)|(x) \leq \theta_n^p(x) |\alpha|(x)$; hence

$$\|E_T^n \alpha\|_1 \leq \int_X \theta_n^p(x) |\alpha|(x) dV \leq A_n \|\alpha\|_\infty, \quad (3.1)$$

where

$$A_n = \max \left\{ \int_X \theta_n^p(x) dV, 0 \leq p \leq m \right\}.$$

Let $T^*: H^*(X, R) \rightarrow H^*(X, R)$ denote the induced map on de Rham cohomology. If α is a harmonic form corresponding to the eigenvalue β of T^* , $E_T^n \alpha = \beta^n \alpha + d\gamma_n$, so by (3.1)

$$\|\beta\|^n \|\alpha\|_1 = \|HE_T^n \alpha\|_1 \leq \|H\|_1 \|E_T^n \alpha\|_1 \leq \Lambda_n \|H\|_1 \|\alpha\|_\infty. \quad (3.2)$$

Now if $h_1 = \limsup_{n \rightarrow \infty} \log \Lambda_n^{1/2}$, (3.2) gives $\log \lambda \leq h_1$, where λ is the spectral radius of T^* .

Remark. The argument also gives the stronger result that for any form α (closed or not), $\limsup_{n \rightarrow \infty} (1/n) \log \|E_T^n \alpha\|_1 \leq h_1$.

4. COMPARISON OF h_1 AND h_2

If V_x denotes the tangent space at a point x in X , $\|DT^n(x)\langle\xi\rangle\|^2$ and $\|\xi\|^2$ can be viewed as quadratic forms on V_x . The first is positive semidefinite and the second is positive definite, so the eigenvalues $\tau_1^2, \tau_2^2, \dots, \tau_m^2$ of the first with respect to the second are defined and can be assumed to satisfy $\tau_1(x) \geq \tau_2(x) \geq \dots \tau_m(x) \geq 0$. It is well known that $\theta_n^p(x)$ as defined above is given by

$$\theta_n^p(x) = \prod_{j=1}^p \tau_j(x).$$

If g denotes the original Riemannian metric on X and g_n the metric defined by $g_n(\xi, \eta) = g(\xi, \eta) + g(DT^n\langle\xi\rangle, DT^n\langle\eta\rangle)$, the volume element dV_n associated to g_n is given by

$$\frac{dV_n}{dV}(x) = \prod_{j=1}^m (1 + \tau_j(x)).$$

If the right side is multiplied out, one of the terms is $\theta_n^p(x)$, hence

$$V(X)A_n \leq V_n(X). \quad (4.1)$$

Now if $\gamma_n(\xi, \eta) = \sum_{j=0}^n g(DT^j\langle\xi\rangle, DT^j\langle\eta\rangle)$ is still another Riemannian metric on X , with associated volume element dU_n , one has $dU_n(x)/dV_n(x) \geq 1$ and by (4.1)

$$V(x)A_n \leq U_n(X). \quad (4.2)$$

It is known [2, Sect. 6] that if a distance function is defined on X corresponding to the metric γ_n , then dU_n is the corresponding Hausdorff measure. This, of course is not the same as the Hausdorff measure M_n defined in Section 2; however, it will be shown that

$$\limsup_{n \rightarrow \infty} (1/n) \log U_n(X) = \limsup_{n \rightarrow \infty} (1/n) \log M_n(X) = h_2. \quad (4.3)$$

To prove (4.3), first observe that if the definition in Section 2 is made with the distance function

$$\delta_n(x, y) = \left(\sum_{j=0}^n d(T^j(x), T^j(y))^2 \right)^{1/2},$$

in place of d_n , the quantity h_2 is unaffected. This follows from

$$(1/n) \delta_n(x, y) \leq d_n(x, y) \leq \delta_n(x, y)$$

and

$$M(n, \epsilon) \leq M^*(n, \epsilon) \leq n^m M(n, \epsilon/n),$$

where $M^*(n, \epsilon)$ is the analog for δ_n of $M(n, \epsilon)$. Moreover, the Riemannian metric γ_n defined above is related to δ_n by

$$\gamma_n(\xi, \xi)^{1/2} = \lim_{t \rightarrow 0} (1/t) \delta_n(x(0), x(t)),$$

where ξ is the tangent vector to $x(t)$ at $x = 0$. It is then easy to verify that the Hausdorff measure corresponding to δ_n is nothing but dU_n . This proves (4.3), which together with (4.2) shows that $h_1 \leq h_2$.

We collect our results:

THEOREM. *Let T be a C^1 map of a compact manifold. Let λ be the spectral radius of the induced map on cohomology and let h be the entropy of T . Then the quantities h_1, h_2, h_3 defined above satisfy $\log \lambda \leq h_1 \leq h_2 \geq h_3$ and $h_3 \leq h$. Moreover under the assumptions of Proposition 2.1, $h_2 = h_3$, hence $\log \lambda \leq h$.*

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